The PRB and Other Potential Successors to the Flawed PRD as a Measure of Vertical Assessment Inequity

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Gloudemans' introduction of the price-related bias (PRB) measure of vertical assessment inequity in *Fair and Equitable* (2011) is a welcome step in finding alternatives to the price-related differential (PRD) as the most commonly used statistical measure. It also appears to have distinct advantages over currently preferred academic alternatives, such as the Clapp measure, although the PRB itself can be refined or further improved under certain circumstances, as described below.

As I have previously noted (Denne 2011), the PRD appears to have arisen as a calculation of convenience to accompany U.S. Census data developed for another purpose, serving only to allow users "to obtain some notion of any association, within a jurisdiction, between assessed values and property sales price ranges" (U.S. Department of Commerce 1957). Searches among notable treatises on assessment equity have failed to uncover any prior usage of the measure before the *1957 Census of Governments* in which it was introduced. As hinted in the quote above, the prior art was to calculate separate assessment-sales ratios for a series of increasing ranges of assessed values or of sale prices.

The PRD has several flaws. As Gloudemans (2011) points out, it does not measure the severity of the reported bias in comprehensible terms. It is usually employed to make a strict yes-or-no compliance decision in a manner akin to testing for statistical significance, rather than testing for the magnitude of an effect. More troublesome is the statistical "errors-in-variables" issue, arising from the fact that market value, the basis of the assessments being tested, is measured with errors in assessments in the numerator and in sale prices in the denominator of assessment-sales ratios. Recognition of this fact led to the development of an asymmetric compliance range for the PRD (0.98 to 1.03), as further discussed by Gloudemans and reflected in various editions of the IAAO *Standard on Ratio Studies* (IAAO 1980, 1990, 1999, 2007). The most troublesome aspect of the PRD, however, is its susceptibility to extreme distortion as a result of heteroscedasticity (inconstant variance among ranges of assessment ratios, especially when the variance is systematically increasing with property value). This flaw was thoroughly described by Jensen (2009).

As a result, several alternatives to the PRD have been developed. In addition to the PRB, which is used in practice by Gloudemans and his colleagues, academics currently favor a test developed by Clapp (1990), as described in table 1.

Table 1. Tests of vertical inequity in property tax assessment (adapted from Sirmans, Diskin and Swint [1995])

Model	Null Hypothesis	Author
$AV = a_0 + a_1 SP$	$a_0 = 0$	Paglin and Fogarty (1972)
$\ln SP = a_0 + a_1 \ln AV$	$a_1 = 1$	Kochin and Parks (1984)
$\ln AV = a_0 + a_1 \ln SP$	$a_1 = 1$	Cheng (1974)
$AV = a_0 + a_1 SP + a_2 SP^2$	$a_0 = a_2 = 0$	Bell (1984)
$AV/SP = a_0 + a_1 SP$	$a_1 = 0$	IAAO (1978)
$\ln SP = a_0 + a_1 \ln AV$, $\ln AV = b_0 + b_1 Z$	$a_1 = 1$	Clapp (1990)
$AV = a_{00} + a_{10} SP + a_{01} LOW$ $+ a_{02} HIGH + a_{11} LOW\ SP + a_{12} HIGH\ SP$	$a_{00} = a_{01}$ $= a_{02} = 0$	Sunderman et al. (1990)

Clapp avoids the errors-in-variables problem by developing what has been called among econometricians an *instrumental variable* (a transformation of available data intended to substitute for unavailable data). Market value, being an expectation (i.e., an average), is unobservable in any particular transaction, which may have a disturbance either up or down. This wreaks havoc on any attempt to explore vertical equity by calculating assessment ratios of ranges of assessed values or sale prices; it is also the reason why the PRB uses value, defined as half of assessed value plus half of sale price for each sale, rather than one or the other, on the right side of the regression equation.

Clapp avoids the problem by defining a new variable called *Z*. If both the assessed value and the sale price for a given property are in the lower third of the available sales data, then *Z* is assigned the value −1. If both the assessment and the sale price are in the upper third of the data, then *Z* is assigned the value 1. For all other observations, *Z* is assigned the value 0. Thus *Z* is an indicator of high- or low-value properties and is used to develop an estimate of the (known) logarithm of assessed value. Such estimates, in turn, are used to derive Clapp's regressivity measure, namely, the slope that characterizes the relationship between the estimated log of assessed value and the log of sale price.

The PRB minimizes the errors-in-variables problem by defining (pseudo) market value as being equal to the sum of half the (time-adjusted) sale price and half the assessed value (assuming the latter is expressed at full, not fractional, value). As noted elsewhere, the PRB is obtained by taking the base-2 logs of the ratios of such (pseudo) values to the median (pseudo) market value and using them to predict the ratio formed by taking each individual assessment-sales ratio, subtracting from it the median ratio, and dividing the result by the median ratio. The coefficient of the regression equation developed by the prediction equation indicates whether assessment-sales ratios tend to be systematically lower, higher, or steady as market values increase. Thus it provides an estimate of the magnitude of any such systematic tendencies (i.e., the effect size) in addition to permitting a calculation to be made of the likelihood of such results occurring by chance alone in the absence of a real systematic difference (i.e., the statistical significance of the finding).

Heteroscedasticity (nonconstant variance of the dependent variable across the range of the independent variables) assumes new importance with Jensen's paper and is treated differently by Clapp and the PRB. Unlike the Clapp measure, in which the independent variable values for the instrumental variable are all either 1 or −1, the dependent variable (and hence the residuals) in the PRB regression may be subject to nonconstant variance over the range of the independent variable, even though the latter has been expressed as a logarithm, which sometimes tends to minimize this tendency. The PRB, though, has a somewhat counterbalancing advantage over the Clapp measure in that it makes use of all the available data, rather than at most two-thirds of it and thus is more sensitive when sample sizes are small.

A systematic exploration of the relative advantages of the PRB and Clapp measure thus seems warranted. Such an undertaking is of course complicated by the unobservability of market value, leaving practitioners with only assessed values and sale prices to work with. For purposes of exploring the characteristics of the two measures, rather than analyzing a particular set of data, simulation studies that approximate real-world circumstances as closely as possible can prove invaluable. I previously reported the initial results of such simulation studies (Denne 2011), which showed that the PRB was far more sensitive than the Clapp measure at detecting known biases in small samples. I also demonstrated that the PRB had a slightly higher rate of false positive findings when assessments are heteroscedastic, as would be expected from Jensen's work.

My report ended with a note that econometricians had developed a number of techniques to address the problem of heteroscedasticity in addition to the instrumental variable approach adopted by Clapp. The most promising of them appeared to be the use of weighted least squares (WLS) regression, instead of ordinary least squares (OLS) regression, to minimize the influence of observations that appear to be most subject to variance and to maximize the influence of observations that appear to be most reliable. The balance of this paper reports on an exploration of the utility of the WLS approach as a refinement of the PRB.

Study Details

Because market values are not directly observable, the only way to study the effects of deviations from them is to define them by fiat in simulations, along with a variety of disturbances from them, for both the observable sale prices and the observable assessments. The following are the specifications employed to obtain the results reported in the following section. My experience in analyzing actual ratio study data from a large number and a large variety of jurisdictions suggests that the simulations approximate reality to a reasonable degree.

Market Value

Market values were specified as random numbers drawn from a mixture of two distributions: 80 percent from a lognormal distribution (i.e., 200,000 × a lognormal distribution, where the underlying normal distribution has a mean of 0 and a variance of 0.5^2) and the remaining 20 percent from a normal distribution (in statistical notation $\sim N\,[50000, 20{,}000^2]$). A histogram of a typical distribution of market values is given in figure 1, panel 1.

Sale Price

Sale prices were specified as being the result of applying random multiplicative disturbances to the market values previously specified. The disturbances were defined to occur in a bow-tie shape, wider at the extremes than in the middle, to reflect the observation that sale prices depart less from market values where market transactions are most numerous, than they do at the extremes. The middle segment was modeled as having normally distributed disturbances, each with a mean of 0 and a standard deviation of 5 (in statistical notation the distribution is $\sim N[0, 52]$). Two different bow-tie disturbances were tested. In one set the flare went from the minimum sale price to the 25th percentile and reappeared from the 85th percentile to the maximum sale value. In the other pattern the flare went from the minimum value to the 15th percentile and reappeared from the 75th percentile to the maximum. In both cases the maximum left-side standard deviation was 15, and the maximum right-side standard deviation was 12; the mean was 0 in all cases. The disturbances defining sales were then multiplied by 1.75 to generate pseudo sale prices. Figure 1, panel 2, illustrates these disturbances.

Assessed Value

Assessed values were specified in much the same way as sale prices, that is, as the result of random multiplicative disturbances from specified market values. The random disturbances that defined pseudo assessments were developed independently from those defining pseudo sale prices. Again, a bow-tie shaped flare was used with the same two patterns of asymmetric flares. The pattern used for sales was consistent with the pattern used for assessments for a two-way, not four-way, test. In order to reflect the observation that market participants may have more, and more timely, information available to them than the assessor, the disturbances defining assessments were given a larger multiplier, 2.25, instead of the 1.75 used for sales. Figure 1, panel 3, illustrates the resulting assessments plotted by the resulting sales; notice the flares at either end, although they are more obvious on the right for high-priced properties.

Assessment Biases

Assessment regressivity biases of three types were tested:

1. No deliberate biases; the only ones present would have arisen as a result of the (unbiased) random disturbances described above, specifically in R language,

   $\text{HypoAsmt} = \text{round}[(1 + \text{DisturbAs}) \times \text{HypoVal}, -2]$

2. A deliberate linear bias in addition to the (unbiased) disturbances described above, modeled as

   $\text{HypoAsmt} = \text{round}[10{,}000 + 0.93 \times (1 + \text{DisturbAs}) \times \text{HypoVal}, -2]$

   (Note: In a later step the rate at which the assessment ratio declined with value was increased by changing the 0.93 to 0.88.)

3. A deliberate nonlinear bias whose relative multiplicative magnitude increased with market value, modeled as

   $\text{HypoAsmt} = \text{round}\big((1 + \text{DisturbAs}) \times \{\text{HypoVal} - [(0.001 \times \text{HypoVal})^{1.75}]\}, -2\big)$.

In addition to three alternatives of equitable assessments and assessments deliberately biased in the two ways described, a further source of bias was considered to reflect a potential source of bias outside the purview of the assessor, which is sometimes argued to occur as a result of imperfections in appeal procedures. An example of this kind of indirect bias is described in the following section.

Assessed Value Adjustments

Assessed values as defined above were tested in three different ways: (1) exactly as described above, (2) modified by a hypothesized omniscient board of appeals with the capacity to reduce assessments that exceed market values by a threshold of 50,000 to the specified market value (not assessed value) of the property, leaving all other assessments unchanged, and (3) modified by the same board of appeals as described above, except that the appeal threshold is 0 instead of 50,000, thus leaving assessment inequities only on the side of underassessments with no over-assessments. Figure 1, panels 4–6, illustrate these alternatives.

Sample Size

Two sample sizes were tested: 100 sales and 1,000 sales, with all complications due to time adjustments, financing, and the like assumed to have been eliminated.

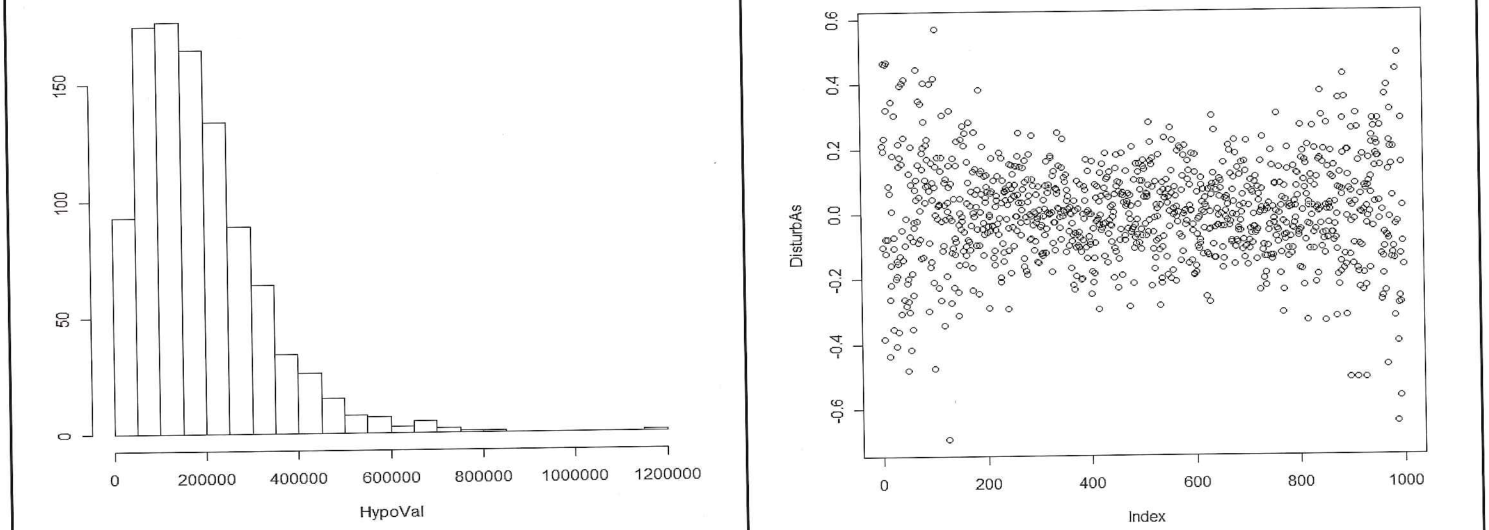
Iteration Count

For all tests, 2,000 iterations were performed. Thus, for a sample of size 1,000, 2 million pseudo market values, 2 million pseudo sale prices, and 2 million pseudo assessments were generated. This process was repeated with different random data for each of the 48 different combinations of settings of the various parameters described above and summarized in table 2.

Heteroscedasticity Adjustment

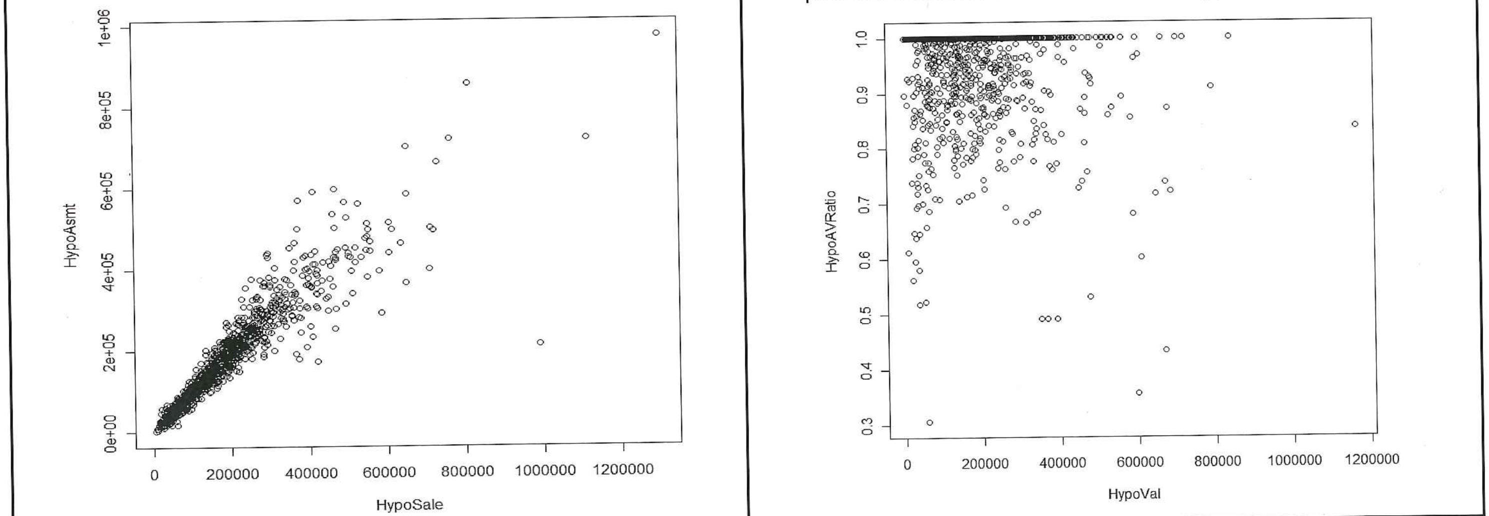
WLS regression, in which the weight

Figure 1. Distribution of a typical set of pseudo market values used in the reported simulations



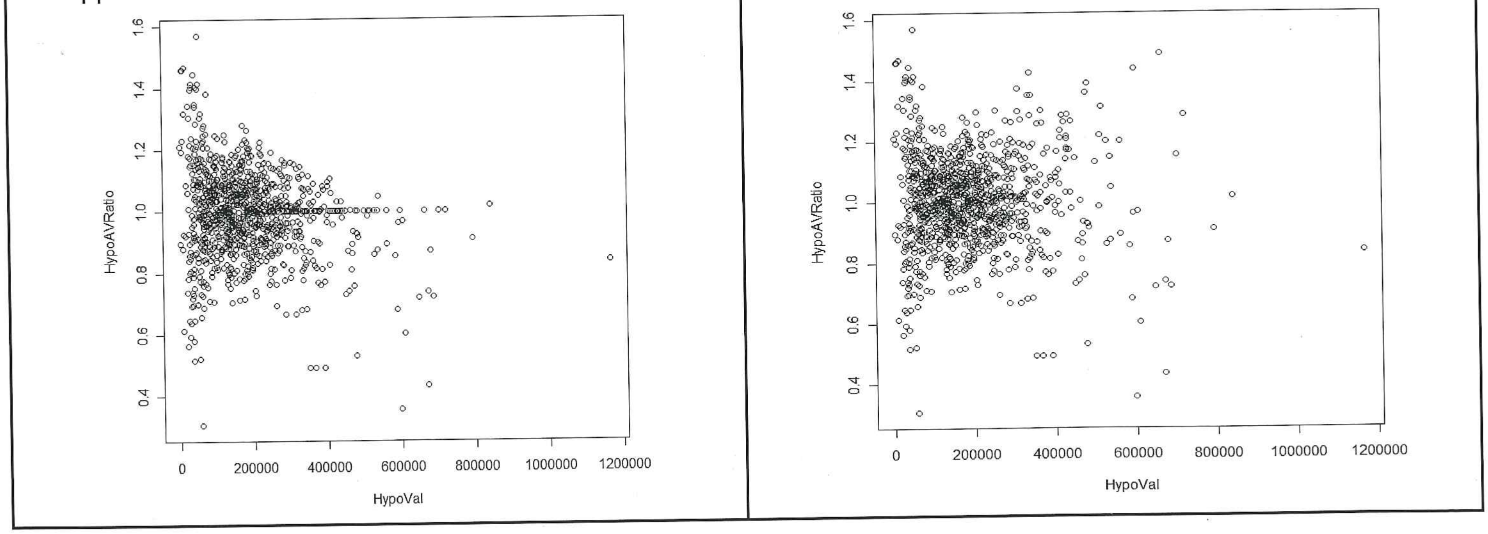
(1) Distribution of hypothetical (unknown) values

(2) Sample distribution of disturbances, bow-tie shape



(3) Typical scatter plot of assessments by sales

(4) Typical scatter plot of assessment-sales ratios by value, zero appeal threshold (under-assessments only; no over-assessments)



(5) Typical scatter plot of assessment-sales ratios by value, appeal threshold = 50,000 (induces bias)

(6) Typical scatter plot of assessment-sales ratios by value, infinite appeal threshold (no adjustments)

Table 2. Rates of positive regressivity findings: Clapp, PRB, and PRBivw, in various scenarios

Hypothesized Configuration Descriptions					Rate (%) of Positive Findings		
Configuration or Line No.	Bias Type	Sample Size	Heteroscedasticity Shape (Bow-tie Percentiles)	Appeal Threshold	Clapp	PRB	PRBivw
1	None	100	.25, .85	0	0.1	5.7	1.4
2	None	100	.25, .85	50,000	0.1	16.9	6.3
3	None	100	.25, .85	infinite	0.6	11.9	3.9
4	None	100	.15, .75	0	0.0	4.6	1.0
5	None	100	.15, .75	50,000	0.1	14.2	3.7
6	None	100	.15, .75	infinite	0.3	10.6	2.9
7	None	1,000	.25, .85	0	1.1	0.1	0.0
8	None	1,000	.25, .85	50,000	0.6	0.8	0.0
9	None	1,000	.25, .85	infinite	1.5	0.2	0.0
10	None	1,000	.15, .75	0	0.0	0.0	0.0
11	None	1,000	.15, .75	50,000	1.7	0.6	0.0
12	None	1,000	.15, .75	infinite	1.1	0.0	0.0
13	Linear.93	100	.25, .85	0	0.1	23.8	17.4
14	Linear.93	100	.25, .85	50,000	28.6	94.5	83.8
15	Linear.93	100	.25, .85	infinite	18.1	88.1	70.6
16	Linear.93	100	.15, .75	0	0.3	23.8	17.0
17	Linear.93	100	.15, .75	50,000	29.5	94.8	86.9
18	Linear.93	100	.15, .75	infinite	16.0	85.3	72.6
19	Linear.93	1,000	.25, .85	0	40.6	4.1	0.3
20	Linear.93	1,000	.25, .85	50,000	100.0	100.0	100.0
21	Linear.93	1,000	.25, .85	infinite	100.0	100.0	97.4
22	Linear.93	1,000	.15, .75	0	79.0	3.0	0.1
23	Linear.93	1,000	.15, .75	50,000	100.0	100.0	100.0
24	Linear.93	1,000	.15, .75	infinite	100.0	100.0	98.0
25	Powr1.75	100	.25, .85	0	0.0	9.2	5.0
26	Powr1.75	100	.25, .85	50,000	0.6	27.4	17.0
27	Powr1.75	100	.25, .85	infinite	0.4	23.6	13.2
28	Powr1.75	100	.15, .75	0	0.0	9.0	4.1
29	Powr1.75	100	.15, .75	50,000	0.5	24.6	12.6
30	Powr1.75	100	.15, .75	infinite	0.3	18.0	8.2
31	Powr1.75	1,000	.25, .85	0	0.2	0.2	0.1
32	Powr1.75	1,000	.25, .85	50,000	30.0	9.1	0.3
33	Powr1.75	1,000	.25, .85	infinite	13.9	4.2	0.1
34	Powr1.75	1,000	.15, .75	0	2.1	0.1	0.0
35	Powr1.75	1,000	.15, .75	50,000	38.9	5.0	0.1
36	Powr1.75	1,000	.15, .75	infinite	10.3	0.9	10.3
37	Linear.88	100	.25, .85	0	0.5	33.4	27.6
38	Linear.88	100	.25, .86	50,000	28.0	94.3	82.7
39	Linear.88	100	.25, .87	infinite	22.3	90.8	75.8
40	Linear.88	100	.15, .75	0	0.6	33.1	30.0
41	Linear.88	100	.15, .76	50,000	27.9	93.9	86.3
42	Linear.88	100	.15, .77	infinite	19.4	88.9	78.8
43	Linear.88	1,000	.25, .85	0	80.4	16.9	3.8
44	Linear.88	1,000	.25, .86	50,000	100.0	100.0	99.9
45	Linear.88	1,000	.25, .87	infinite	100.0	100.0	99.2
46	Linear.88	1,000	.15, .75	0	96.0	14.8	6.2
47	Linear.88	1,000	.15, .76	50,000	100.0	100.0	100.0
48	Linear.88	1,000	.15, .77	infinite	100.0	100.0	99.5

assigned to each observation in the regression is inversely proportional to the variance of the residuals in the neighborhood of the observation, was adopted here to address the problem of heteroscedasticity described earlier. Some authors claim that WLS affects only the confidence intervals generated for the regression coefficients (and hence the findings of their significance) and not the coefficients themselves, but that claim is not true. Both the coefficients and their associated standard errors (or significance levels) are affected. In standard econometric usage it is most commonly the case that the variance of the residuals increases with the dependent variable (market value in the case of the PRB). In assessment ratio studies, however, the opposite is often the case, because of the high leverage that results from creating a ratio with a denominator that is subject to error. This is especially the case in the presence of an omniscient board of appeals with a nontrivial appeal threshold.

Thus the problem here is more difficult and perhaps less susceptible to the tested remedy. Some statistical software is capable of developing optimal weights for the WLS algorithm automatically (IBM® SPSS® software is one; SAS reputedly has such capability; something similar could also be programmed in R language, although to my knowledge there is nothing ready-made for this purpose in R). The tests reported here, however, used the simpler procedure of weighting on the basis of the inverses of the bow-tie shaped disturbances. Such details, of course, are unknown in practice, but the objective here is to test the theory before attempting to refine all the details.

Performance Measures

Since Jensen's criticism of the PRD is essentially that it generates false positive findings to an unacceptable degree, determining the false positive rates of the PRB, the PRB with inverse variance weighting (PRBivw), and the Clapp measure under varying circumstances are major objectives of this study. Of comparable interest, however, are indications of the power or sensitivity of the alternative tests in the presence of known biases under various circumstances. Of somewhat secondary interest is how the various test data sets perform when measured by using the standard ratio study statistics, such as

Table 3. Standard ratio study statistics for the simulated data in the various scenarios

Configuration or Line No.	Coefficient of Dispersion (COD)						Price-Related Differential (PRD)						Spearman Rank Correlation Coefficient					
	Min	Q25	Median	Mean	Q75	Max	Min	Q25	Median	Mean	Q75	Max	Min	Q25	Median	Mean	Q75	Max
1	8.6	11.5	12.3	12.4	13.2	26.3	0.97	1.01	1.02	1.02	1.03	1.16	−0.494	−0.147	−0.069	−0.072	0.003	0.293
2	11.3	14.3	15.3	15.4	16.3	29.5	0.98	1.02	1.03	1.03	1.05	1.18	−0.513	−0.227	−0.152	−0.151	−0.076	0.264
3	12.0	15.0	16.0	16.1	17.1	29.6	0.95	1.01	1.02	1.02	1.03	1.16	−0.498	−0.158	−0.084	−0.081	−0.005	0.350
4	8.1	11.3	12.0	12.1	12.8	22.8	0.98	1.01	1.02	1.02	1.03	1.13	−0.494	−0.190	−0.114	−0.116	−0.041	0.259
5	10.7	13.8	14.7	14.8	15.6	26.6	0.98	1.02	1.03	1.04	1.05	1.16	−0.524	−0.248	−0.175	−0.174	−0.100	0.265
6	11.6	14.7	15.6	15.7	16.6	27.4	0.95	1.00	1.02	1.02	1.03	1.13	−0.511	−0.160	−0.085	−0.084	−0.010	0.376
7	11.2	12.1	12.4	12.4	12.6	17.9	1.00	1.01	1.02	1.02	1.02	1.07	−0.189	−0.095	−0.071	−0.072	−0.048	0.041
8	14.0	15.0	15.3	15.3	15.6	20.9	1.01	1.03	1.03	1.03	1.04	1.09	−0.264	−0.172	−0.148	−0.148	−0.125	−0.027
9	14.6	15.7	16.0	16.0	16.3	21.5	0.99	1.01	1.02	1.02	1.02	1.07	−0.213	−0.105	−0.079	−0.080	−0.057	0.055
10	10.8	11.8	12.0	12.0	12.3	17.6	1.00	1.02	1.02	1.02	1.03	1.07	−0.230	−0.139	−0.116	−0.117	−0.094	−0.008
11	13.4	14.4	14.7	14.7	15.0	20.1	1.01	1.03	1.03	1.04	1.04	1.09	−0.284	−0.195	−0.172	−0.172	−0.150	−0.063
12	14.2	15.3	15.6	15.6	15.9	20.9	0.99	1.01	1.02	1.02	1.02	1.07	−0.217	−0.107	−0.083	−0.084	−0.061	0.050
13	8.8	11.2	11.9	12.0	12.8	27.5	0.99	1.02	1.03	1.04	1.05	1.20	−0.620	−0.297	−0.222	−0.225	−0.150	0.118
14	12.1	15.4	16.6	17.0	18.1	34.5	1.02	1.06	1.08	1.08	1.09	1.27	−0.684	−0.464	−0.399	−0.398	−0.336	−0.019
15	12.0	15.8	17.0	17.3	18.6	34.9	1.00	1.05	1.07	1.07	1.08	1.25	−0.684	−0.422	−0.356	−0.355	−0.291	0.010
16	8.5	10.9	11.7	11.8	12.5	23.5	0.99	1.03	1.04	1.04	1.05	1.16	−0.645	−0.339	−0.267	−0.269	−0.199	0.079
17	11.7	14.9	16.1	16.4	17.4	32.4	1.02	1.06	1.08	1.08	1.09	1.25	−0.712	−0.496	−0.434	−0.432	−0.371	−0.088
18	12.0	15.4	16.5	16.8	17.9	32.1	1.00	1.05	1.06	1.07	1.08	1.22	−0.700	−0.440	−0.374	−0.372	−0.309	−0.010
19	10.9	11.7	12.0	12.0	12.2	17.2	1.02	1.03	1.03	1.04	1.04	1.09	−0.341	−0.250	−0.228	−0.227	−0.205	−0.118
20	14.8	16.5	16.9	16.9	17.4	22.5	1.05	1.07	1.08	1.08	1.08	1.14	−0.504	−0.422	−0.402	−0.401	−0.381	−0.285
21	15.1	16.8	17.2	17.3	17.7	22.8	1.04	1.06	1.07	1.07	1.07	1.12	−0.481	−0.380	−0.359	−0.359	−0.339	−0.242
22	10.6	11.5	11.7	11.7	12.0	16.8	1.02	1.03	1.04	1.04	1.04	1.09	−0.379	−0.295	−0.273	−0.273	−0.252	−0.158
23	14.3	15.9	16.3	16.3	16.7	21.7	1.06	1.07	1.08	1.08	1.08	1.14	−0.541	−0.457	−0.438	−0.437	−0.419	−0.329
24	14.7	16.3	16.7	16.8	17.2	22.1	1.04	1.06	1.07	1.07	1.07	1.12	−0.497	−0.398	−0.378	−0.378	−0.358	−0.264
25	9.3	12.4	13.2	13.3	14.1	27.5	0.98	1.01	1.03	1.03	1.04	1.17	−0.537	−0.218	−0.143	−0.143	−0.068	0.245
26	11.5	14.9	15.9	16.0	17.0	30.3	0.98	1.03	1.04	1.04	1.06	1.19	−0.596	−0.287	−0.214	−0.212	−0.137	0.243
27	12.0	15.1	16.1	16.3	17.3	30.6	0.97	1.02	1.04	1.04	1.05	1.19	−0.596	−0.263	−0.191	−0.188	−0.114	0.245
28	8.8	12.1	12.9	13.0	13.8	24.0	0.98	1.02	1.03	1.03	1.04	1.14	−0.545	−0.254	−0.179	−0.181	−0.106	0.208
29	11.3	14.3	15.2	15.3	16.2	27.4	0.99	1.03	1.04	1.04	1.06	1.18	−0.617	−0.302	−0.230	−0.229	−0.158	0.244
30	11.9	14.7	15.6	15.8	16.7	27.8	0.97	1.02	1.04	1.04	1.05	1.16	−0.608	−0.269	−0.195	−0.194	−0.121	0.266
31	12.1	13.0	13.3	13.3	13.6	18.4	1.01	1.02	1.03	1.03	1.03	1.08	−0.261	−0.167	−0.144	−0.145	−0.121	-0.026
32	14.4	15.6	15.9	15.9	16.2	21.0	1.02	1.04	1.04	1.04	1.05	1.09	−0.325	−0.234	−0.211	−0.211	−0.189	-0.082
33	14.7	15.9	16.2	16.2	16.5	21.3	1.01	1.03	1.04	1.04	1.04	1.09	−0.315	−0.212	−0.188	−0.189	−0.166	-0.063
34	11.7	12.7	13.0	13.0	13.2	18.0	1.01	1.03	1.03	1.03	1.04	1.08	−0.296	−0.205	−0.183	−0.183	−0.162	-0.068
35	14.0	14.9	15.2	15.2	15.5	20.2	1.02	1.04	1.04	1.04	1.05	1.09	−0.345	−0.251	−0.229	−0.229	−0.208	-0.118
36	14.3	15.4	15.6	15.7	16.0	20.6	1.01	1.03	1.04	1.04	1.04	1.08	−0.324	−0.218	−0.195	−0.195	−0.173	-0.069
37	9.3	12.0	12.8	12.9	13.7	28.7	0.99	1.03	1.04	1.04	1.05	1.20	−0.625	−0.332	−0.258	−0.260	−0.187	0.084
38	11.9	15.6	16.9	17.2	18.4	35.0	1.02	1.06	1.07	1.08	1.09	1.27	−0.693	−0.455	−0.391	−0.389	−0.326	−0.014
39	12.1	15.9	17.1	17.5	18.7	35.3	1.00	1.05	1.07	1.07	1.09	1.26	−0.693	−0.434	−0.368	−0.367	−0.304	−0.014
40	8.9	11.8	12.5	12.6	13.4	24.7	1.00	1.03	1.04	1.04	1.05	1.17	−0.656	−0.371	−0.300	−0.301	−0.231	0.045
41	11.8	15.1	16.3	16.6	17.7	32.6	1.01	1.06	1.07	1.08	1.09	1.25	−0.715	−0.483	−0.419	−0.417	−0.356	−0.038
42	12.0	15.4	16.6	17.0	18.1	32.5	1.00	1.05	1.07	1.07	1.08	1.22	−0.705	−0.453	−0.386	−0.385	−0.322	−0.031
43	11.6	12.6	12.8	12.9	13.1	17.9	1.02	1.04	1.04	1.04	1.04	1.09	−0.375	−0.286	−0.264	−0.263	−0.241	−0.153
44	15.1	16.7	17.1	17.2	17.6	22.8	1.05	1.07	1.08	1.08	1.08	1.13	−0.500	−0.413	−0.393	−0.392	−0.373	−0.274
45	15.2	16.9	17.4	17.4	17.8	23.0	1.04	1.07	1.07	1.07	1.08	1.13	−0.492	−0.393	−0.372	−0.372	−0.351	−0.254
46	11.4	12.3	12.6	12.6	12.8	17.5	1.02	1.04	1.04	1.04	1.05	1.10	−0.411	−0.327	−0.306	−0.306	−0.286	−0.192
47	14.5	16.2	16.6	16.6	17.1	22.0	1.05	1.07	1.08	1.08	1.08	1.13	−0.526	−0.441	−0.422	−0.421	−0.402	−0.315
48	14.8	16.4	16.9	16.9	17.3	22.3	1.04	1.06	1.07	1.07	1.08	1.12	−0.509	−0.412	−0.391	−0.391	−0.371	−0.279

the coefficient of dispersion (COD), and other occasionally seen measures, including the rank correlation coefficient. These are summarized in table 3, in which the line numbers (simulation configurations) are the same as those in table 2. I can provide additional detail, upon request, along with the R-code used to obtain the reported simulations, which could easily be modified to test other sets of circumstances if desired.

Table 3. *(continued)*

Configuration or Line No.	Clapp Coefficient						Price-Related Bias (PRB) Coefficient						PRB with Inverse Variance Weighting (PRBivw)					
	Min	Q25	Median	Mean	Q75	Max	Min	Q25	Median	Mean	Q75	Max	Min	Q25	Median	Mean	Q75	Max
1	0.90	0.97	0.99	0.99	1.01	1.10	−0.328	−0.016	0.001	−0.001	0.018	0.069	−0.140	−0.017	−0.003	−0.004	0.009	0.059
2	0.90	0.99	1.01	1.01	1.04	1.14	−0.354	−0.039	−0.017	−0.021	0.002	0.073	−0.148	−0.028	−0.012	−0.013	0.003	0.065
3	0.88	0.97	0.99	0.99	1.02	1.13	−0.321	−0.024	−0.002	−0.005	0.019	0.087	−0.123	−0.013	0.004	0.002	0.019	0.087
4	0.91	0.98	1.00	1.00	1.02	1.10	−0.268	−0.019	−0.002	−0.004	0.013	0.063	−0.114	−0.017	−0.005	−0.006	0.006	0.049
5	0.91	0.99	1.02	1.02	1.04	1.13	−0.341	−0.036	−0.017	−0.020	0.002	0.069	−0.140	−0.024	−0.010	−0.011	0.004	0.058
6	0.88	0.97	0.99	0.99	1.01	1.12	−0.301	−0.018	0.003	0.000	0.022	0.087	−0.113	−0.009	0.007	0.006	0.021	0.084
7	0.96	0.98	0.99	0.99	1.00	1.02	−0.079	−0.007	−0.001	−0.001	0.005	0.027	−0.038	−0.008	−0.004	−0.004	0.000	0.017
8	0.97	1.00	1.01	1.01	1.02	1.06	−0.098	−0.027	−0.019	−0.020	−0.013	0.014	−0.050	−0.017	−0.012	−0.012	−0.007	0.014
9	0.95	0.98	0.99	0.99	1.00	1.04	−0.081	−0.011	−0.004	−0.005	0.003	0.029	−0.033	−0.002	0.003	0.003	0.008	0.030
10	0.97	1.00	1.00	1.00	1.01	1.04	−0.043	−0.009	−0.003	−0.004	0.002	0.025	−0.026	−0.009	−0.006	−0.006	−0.002	0.014
11	0.98	1.01	1.02	1.02	1.02	1.06	−0.073	−0.025	−0.019	−0.019	−0.012	0.012	−0.035	−0.014	−0.010	−0.010	−0.005	0.015
12	0.95	0.98	0.99	0.99	1.00	1.04	−0.049	−0.005	0.001	0.001	0.008	0.031	−0.021	0.002	0.006	0.006	0.011	0.031
13	0.95	1.02	1.03	1.04	1.05	1.15	−0.390	−0.049	−0.031	−0.034	−0.016	0.030	−0.175	−0.045	−0.032	−0.033	−0.019	0.025
14	0.99	1.08	1.11	1.11	1.14	1.27	−0.537	−0.143	−0.108	−0.120	−0.081	−0.004	−0.287	−0.096	−0.076	−0.079	−0.058	0.017
15	0.98	1.07	1.10	1.10	1.12	1.27	−0.509	−0.132	−0.097	−0.108	−0.069	0.007	−0.267	−0.087	−0.066	−0.069	−0.048	0.026
16	0.96	1.03	1.04	1.04	1.06	1.16	−0.310	−0.048	−0.032	−0.035	−0.017	0.026	−0.145	−0.045	−0.033	−0.034	−0.022	0.018
17	1.00	1.09	1.11	1.12	1.14	1.26	−0.532	−0.139	−0.106	−0.117	−0.080	−0.006	−0.247	−0.095	−0.077	−0.079	−0.060	0.008
18	0.98	1.07	1.09	1.10	1.12	1.26	−0.497	−0.123	−0.090	−0.101	−0.064	0.013	−0.226	−0.085	−0.065	−0.068	−0.049	0.020
19	1.00	1.03	1.04	1.04	1.04	1.07	−0.121	−0.040	−0.033	−0.034	−0.028	−0.009	−0.068	−0.036	−0.032	−0.032	−0.028	−0.013
20	1.07	1.10	1.11	1.11	1.12	1.16	−0.224	−0.133	−0.119	−0.122	−0.108	−0.073	−0.123	−0.085	−0.078	−0.079	−0.072	−0.049
21	1.05	1.09	1.10	1.10	1.11	1.15	−0.212	−0.122	−0.108	−0.110	−0.096	−0.056	−0.113	−0.075	−0.068	−0.069	−0.062	−0.039
22	1.02	1.04	1.04	1.05	1.05	1.08	−0.079	−0.039	−0.034	−0.034	−0.029	−0.011	−0.051	−0.037	−0.034	−0.034	−0.030	−0.017
23	1.08	1.11	1.12	1.12	1.12	1.16	−0.194	−0.130	−0.116	−0.119	−0.105	−0.073	−0.109	−0.084	−0.078	−0.079	−0.073	−0.053
24	1.05	1.09	1.10	1.10	1.10	1.15	−0.175	−0.114	−0.101	−0.103	−0.090	−0.050	−0.099	−0.073	−0.067	−0.067	−0.061	−0.040
25	0.91	0.98	1.01	1.01	1.03	1.12	−0.341	−0.028	−0.010	−0.013	0.007	0.063	−0.149	−0.029	−0.015	−0.015	−0.001	0.057
26	0.91	1.01	1.03	1.03	1.06	1.17	−0.380	−0.053	−0.031	−0.035	−0.011	0.066	−0.164	−0.044	−0.027	−0.028	−0.011	0.061
27	0.91	1.00	1.02	1.03	1.05	1.17	−0.371	−0.048	−0.025	−0.029	−0.004	0.068	−0.158	−0.039	−0.022	−0.023	−0.005	0.065
28	0.92	1.00	1.02	1.02	1.04	1.13	−0.289	−0.030	−0.013	−0.015	0.003	0.056	−0.129	−0.028	−0.016	−0.017	−0.004	0.046
29	0.92	1.01	1.04	1.04	1.06	1.16	−0.368	−0.050	−0.029	−0.033	−0.010	0.061	−0.159	−0.039	−0.024	−0.025	−0.009	0.054
30	0.91	1.00	1.02	1.03	1.05	1.16	−0.343	−0.041	−0.020	−0.024	−0.001	0.067	−0.145	−0.033	−0.017	−0.018	−0.002	0.062
31	0.97	1.00	1.01	1.01	1.01	1.05	−0.093	−0.018	−0.012	−0.012	−0.006	0.013	−0.051	−0.020	−0.015	−0.015	−0.011	0.007
32	0.99	1.02	1.03	1.03	1.04	1.08	−0.115	−0.041	−0.033	−0.034	−0.026	0.002	−0.066	−0.033	−0.028	−0.027	−0.022	0.000
33	0.98	1.02	1.03	1.03	1.03	1.08	−0.109	−0.035	−0.027	−0.028	−0.020	0.007	−0.061	−0.028	−0.022	−0.022	−0.017	0.006
34	0.99	1.01	1.02	1.02	1.03	1.06	−0.054	−0.020	−0.014	−0.014	−0.009	0.011	−0.038	−0.020	−0.016	−0.016	−0.012	0.006
35	1.00	1.03	1.04	1.04	1.04	1.08	−0.088	−0.038	−0.031	−0.031	−0.024	0.001	−0.050	−0.029	−0.024	−0.024	−0.019	0.000
36	0.99	1.02	1.03	1.02	1.03	1.07	−0.076	−0.029	−0.022	−0.023	−0.015	0.010	−0.045	−0.022	−0.017	−0.017	−0.012	0.008
37	0.96	1.03	1.05	1.05	1.07	1.16	−0.402	−0.057	−0.039	−0.042	−0.022	0.026	−0.181	−0.051	−0.037	−0.039	−0.025	0.027
38	0.98	1.08	1.11	1.11	1.14	1.28	−0.550	−0.145	−0.109	−0.121	−0.082	−0.003	−0.286	−0.097	−0.076	−0.079	−0.058	0.018
39	0.98	1.08	1.10	1.10	1.13	1.28	−0.531	−0.139	−0.102	−0.114	−0.074	0.002	−0.278	−0.092	−0.071	−0.074	−0.052	0.022
40	0.96	1.03	1.05	1.05	1.07	1.17	−0.321	−0.056	−0.039	−0.042	−0.024	0.022	−0.153	−0.052	−0.040	−0.041	−0.029	0.017
41	0.99	1.09	1.11	1.11	1.14	1.27	−0.543	−0.140	−0.105	−0.117	−0.079	−0.005	−0.246	−0.095	−0.076	−0.079	−0.060	0.009
42	0.98	1.08	1.10	1.10	1.13	1.27	−0.518	−0.130	−0.095	−0.107	−0.069	0.009	−0.236	−0.089	−0.069	−0.072	−0.053	0.015
43	1.01	1.04	1.05	1.05	1.05	1.08	−0.132	−0.047	−0.041	−0.042	−0.035	−0.015	−0.077	−0.042	−0.038	−0.038	−0.034	−0.018
44	1.06	1.10	1.11	1.11	1.12	1.16	−0.227	−0.135	−0.121	−0.123	−0.109	−0.072	−0.124	−0.085	−0.078	−0.078	−0.072	−0.047
45	1.05	1.10	1.10	1.10	1.11	1.16	−0.220	−0.128	−0.114	−0.116	−0.102	−0.060	−0.119	−0.080	−0.073	−0.074	−0.067	−0.043
46	1.02	1.05	1.05	1.06	1.06	1.09	−0.088	−0.047	−0.041	−0.042	−0.036	−0.017	−0.059	−0.045	−0.041	−0.041	−0.037	−0.022
47	1.07	1.11	1.11	1.11	1.12	1.16	−0.196	−0.130	−0.117	−0.119	−0.106	−0.069	−0.111	−0.084	−0.078	−0.078	−0.072	−0.050
48	1.06	1.09	1.10	1.10	1.11	1.15	−0.184	−0.120	−0.107	−0.109	−0.095	−0.054	−0.104	−0.078	−0.072	−0.072	−0.066	−0.045

Results

Adding inverse variance weighting to the PRB described by Gloudemans reduces, but does not guarantee, the elimination of the problem of false positives arising from heteroscedasticity. Both the PRB and the PRB with inverse variance weighting (PRBivw) are far more sensitive than the Clapp measure in detecting vertical inequity when sample sizes are on the order of 100, which is fairly large for many local ratio studies, but small in the context of this study. The Clapp measure's lack of power is probably attributable to the fact that it uses at most two-thirds of the data available to it and, in practice, substantially less. More detailed findings are described below.

For the test sets with no deliberate bias, the PRBivw substantially reduces the rate of false positives exhibited by the PRB, as shown on lines 1–12 of table 2. For an infinite threshold (i.e., there is no board of appeals adjustment), the rate of positive findings falls to 3–4 percent for the PRBivw from the 11–12 percent rates of the PRB when sample sizes are 100. (A positive finding for either the PRB or the PRBivw is defined here as a slope coefficient with an absolute magnitude of at least 5 percent for each value doubling, combined with a confidence level of at least 95 percent that the coefficient is different from 0. The rate of false positives could also be adjusted by changing either or both of these criteria, but such changes were outside the scope of this investigation.) For sample sizes of 1,000, as on lines 7–12 of table 2, the rate of false positives is not problematic for any of the measures, although there is some suggestion that the Clapp measure may have a slight disadvantage relative to the other two for larger sample sizes.

As an interesting side note, comparing results for an infinite-appeal threshold to a 50,000-appeal threshold, the most realistic of the simulations, lines 2, 3, 5, and 6 (in table 2) suggest that, when measured by the PRBivw, such appeal adjustments add regressivity on the order of 1–2 percent for small samples (or 3–4 percent if measured by the PRB) and almost nothing for larger samples.

> ***When assessment ratio data are subject to differing variability that changes systematically with value, analysts may be well advised to refine the PRB using a WLS procedure and weights that vary inversely with the variability of the ratios. The optimum way of doing so remains an area for further research.***

The power, or sensitivity, of the measures is of equal, if not greater, interest compared to the rate of false positives. The PRB and the PRBivw, which trails the PRB slightly, have a substantial advantage over the Clapp measure for small samples in the most realistic circumstances, as shown on lines 14 and 17 (in table 2), where they detect 94–95 percent and 84–87 percent of the cases of bias, respectively, compared to the detection rate of 29–30 percent for the Clapp measure.

For larger sample sizes and linear biases, there is no appreciable difference among the three measures under the most realistic circumstances, as shown on lines 20 and 23. The marked differences between the measures for linear biases when the appeal threshold is nonexistent, shown on lines 13, 16, 19, and 22, is of uncertain practical significance, because it arises in a highly unrealistic set of circumstances—only under-assessments with no over-assessments, as illustrated in the scatter plot shown in figure 1, panel 4. To further test the pattern of power or sensitivity of the tests, the linear rate of deliberate linear bias was raised, by changing the rate of assessment decline to 88 percent from the previous 93 percent. Comparing lines 14, 17, 20, and 23 with lines 38, 41, 44, and 47 for the most realistic settings (50,000 appeal threshold with both small and large samples and both alternatives for the disturbances) reveals that there are no appreciable differences in the sensitivities of the measures as a consequence of the more pronounced linear bias.

For the modeled nonlinear bias, the results were somewhat the same as for the linear bias, with one surprising exception. As with the linear bias, the PRB and the PRBivw were substantially more powerful than the Clapp measure in small samples for realistic circumstances. However, the Clapp measure sharply outperformed both the PRB and the PRBivw for larger samples under these circumstances, and the relative performances of the PRB and PRBivw differed, with the latter being essentially powerless and the former stronger when the bow-tie shaped disturbance was larger for low-value properties than for high-value ones. When there was no omniscient board of appeals effect (all biases were as designed), the PRBivw and Clapp measure tied for best when the larger bow-tie flare was for high-value properties. When the flare was greater to the left than to the right, the power of PRBivw became negligible, but the PRB gained a modicum of power, although it still trailed that of the Clapp measure. The practical significance of these anomalous findings is not yet fully understood.

In general, the PRBivw is a worthwhile refinement of the PRB in reducing (although not by itself eliminating, at least with the weighting tested) the rate of false positives. This reduction comes at the cost of some moderate loss of power in detecting regressive biases when they are known to exist in a linear fashion. The rate of that linear bias appears not to have much bearing on the difference. It is worth emphasizing, however, that even with the noted loss of power or sensitivity, both the PRB and the PRBivw outperform the alternative Clapp measure in such circumstances. When samples are large and biases are increasing with value, the relative power of the two is somewhat uncertain, perhaps a subject for future research, along with the optimal means of developing weights for WLS in the absence of the perfect knowledge assumed in this investigation.

The results reported for standard ratio study statistics in table 3, calculated from the test sets of pseudo sales and assessments reported here, appear unexceptionable, lending some credence to the hypothesis that these results, obtained from simulated data, may prove useful in the real world. When assessment ratio data are subject to differing variability that changes systematically with value, analysts may be well advised to refine the PRB using a WLS procedure and weights that vary inversely with the variability of the ratios. The optimum way of doing so remains an area for further research.

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